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Effect of relaxation parameters on the quantum theory of a detuned laser with inhomogeneous broadening

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Abstract. The results of the quantum theory of a laser given by Riska and Stenholm for the case of inhomogeneous broadening and detuning are re-derived by avoiding the Doppler limit approximation to show the exact influence of relaxation parameters on the theory. The threshold condition is seen to be dependent on γ_{ab}/ku where γ_{ab} is the relaxation parameter and ku the Doppler parameter. Its value is found to be higher than earlier results. It is noted that the photon distribution function at resonance is much closer to the corresponding distribution obtained from exact calculations for the tuned case. The photon number at the peak of the distribution curve is much lower than earlier values for comparable situations.

1. Introduction

Lamb's theory for gas lasers (Lamb 1964) which explains the operations of gas lasers fairly well is found to be a quantitatively accurate theory away from threshold (Riska and Stenholm 1970a). However, due to the approximations involved, the results of his theory are valid only for low intensities and the Doppler limit (Stenholm and Lamb 1969). The present authors have already shown that Lamb's theory is more useful when the exact ratio of the Doppler parameter ku to the relaxation parameter γ_{ab} is properly considered (Mohanty and Nayak 1974†) though this extension does not remove the shortcomings of the Lamb theory near threshold.

The quantum nature of the electromagnetic field has been incorporated within the framework of semiclassical theory by Scully and Lamb (1967) who have limited themselves to the case of stationary atoms. Kim *et al* (1970) considered the case of atomic motion but their treatment is more involved as they have included the effect of atomic recoil due to photon absorption or emission. Riska and Stenholm (1970a, b), while considering the case of moving atoms, have limited their treatment to the Doppler approximation. Thus their theory is not capable of explaining the exact effect of the term γ_{ab}/ku on the operation of a gas laser. The effect of this term is more pronounced on a detuned laser (Mohanty and Nayak 1974†). This paper primarily aims to study the effect of moving atoms on the operation of a gas laser when the Doppler approximation is relaxed.

2. The photon distribution

We use the model of Riska and Stenholm (1970b) as the essential difference between their work and the present work occurs only when the velocity distribution is taken into

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account. The laser action takes place between an upper (a) level and a lower (b) level with a positive transition frequency ω. The loss mechanism is incorporated by introducing the atoms into the lower one of two broad levels α and β which rapidly decay into two lower levels with the rates γ_α and γ_β.

The diagonal elements ρ_{nn} of the density matrix for the coupled field–atom system denote the photon density for a photon number n. The equation of motion for ρ_{nn}, which is the same as equation (13) of Riska and Stenholm (1970b) is given by

$$\begin{aligned} \mathrm{d}\rho_{nn}/\mathrm{d}t = & -g^2(n+1)\gamma_{ab}r_a\gamma_a^{-1}\left(\int \mathrm{d}v\, W(v)(A_+ + A_-)\right. \\ & \left. -2g^2(n+1)\gamma_{ab}^2(\gamma_a\gamma_b)^{-1}\int \mathrm{d}v\, W(v)(A_+ + A_{-1})^2\right)\rho_{nn} \\ & + \text{the terms repeated with } n \text{ replaced by } (n-1). \end{aligned} \tag{1}$$

Here γ_η are the decay terms from state η, $\gamma_{ab} = \frac{1}{2}(\gamma_a + \gamma_b)$, r_η is the total average injection rate of atoms into the state η, g denotes the coupling constant, and

$$A_\pm = [\gamma_{ab}^2 + (kv \pm \Delta)^2]^{-1}$$

where k is the wavenumber for the incident radiation, v is the velocity parameter and $\Delta = (\omega - \Omega)$, Ω being the frequency of the cavity mode.

The velocity distribution, assumed Maxwellian for a gas laser, is given by

$$W(v)\,\mathrm{d}v = (\sqrt{\pi}u)^{-1}\exp(-v^2/u^2)\,\mathrm{d}v. \tag{2}$$

The following integrals occur in equation (1):

$$I_{1\pm} = (\sqrt{\pi}u)^{-1}\int_{-\infty}^{+\infty} \mathrm{d}v\,\exp(-v^2/u^2)[\gamma_{ab}^2 + (kv \pm \Delta)^2]^{-1} \tag{3a}$$

$$I_{2\pm} = (\sqrt{\pi}u)^{-1}\int_{-\infty}^{+\infty} \mathrm{d}v\,\exp(-v^2/u^2)[\gamma_{ab}^2 + (kv \pm \Delta)^2]^{-2} \tag{3b}$$

$$I_3 = 2(\sqrt{\pi}u)^{-1}\int_{-\infty}^{+\infty} \mathrm{d}v\,\exp(-v^2/u^2)[\gamma_{ab}^2 + (kv + \Delta)^2]^{-1}[\gamma_{ab}^2 + (kv - \Delta)^2]^{-1}. \tag{3c}$$

These can be evaluated, giving

$$I_{1+} = I_{1-} = \sqrt{\pi}(ku\gamma_{ab})^{-1}U(z_+) \tag{4a}$$

$$I_{2+} = I_{2-} = \sqrt{\pi}(2\gamma_{ab}ku)^{-2}\left[y^{-1}(w(z_+) + w(z_-)) - \mathrm{i}\left(\frac{\mathrm{d}}{\mathrm{d}z_-}w(z_-) + \frac{\mathrm{d}}{\mathrm{d}z_+}w(z_+)\right)\right] \tag{4b}$$

$$I_3 = \sqrt{\pi}(ku)^{-1}(\Delta^2 + \gamma_{ab}^2)^{-1}[U(z_+)/\gamma_{ab} + V(z_+)/\Delta] \tag{4c}$$

where

$$z_\pm = \pm x + \mathrm{i}y = \pm\Delta/ku + \mathrm{i}\gamma_{ab}/ku$$

and

$$w(z_\pm) = U(z_\pm) + \mathrm{i}V(z_\pm) = \frac{\pi}{\mathrm{i}}\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{-t^2}\,\mathrm{d}t}{z_\pm - t}.$$

The function $w(z)$, known as the Gaussian of the complex variable, has been extensively tabulated (Faddeyeva and Terent'ev 1961).

After adding the linear loss terms in the same way as Riska and Stenholm (1970a) we have:

$$\frac{d\rho_{nn}}{dt}=-A(n+1)\left\{U(x,y)-\frac{n+1}{4}\frac{B}{A}\left[U(x,y)+\frac{y}{2}\left(\frac{\partial V(x,y)}{\partial x}-\frac{\partial U(x,y)}{\partial y}\right)\right.\right.$$
$$\left.\left.+\frac{\gamma_{ab}}{\gamma_{ab}^2+\Delta^2}\left(U(x,y)+\frac{y}{x}V(x,y)\right)\right]\right\}\rho_{nn}+C(n+1)\rho_{n+1,n+1}$$
$$+\text{the same terms with } n \text{ replaced by } (n-1). \tag{5}$$

We retain the definitions of A, B, and C as given by Riska and Stenholm (1970a) for purpose of future comparison:

$$A=\frac{2\sqrt{\pi}g^2r_a}{\gamma_a ku};\qquad B=\frac{4g^2}{\gamma_a\gamma_b}A$$
$$C=2g^2r_\beta\gamma_{\alpha\beta}\int_{-\infty}^{+\infty}\frac{dvW(v)}{\gamma_\beta[\gamma_{\alpha\beta}^2+(kv)^2]}\simeq\frac{2g^2r_\beta}{\gamma_\beta\gamma_{\alpha\beta}}. \tag{6}$$

The steady state solution of this equation is:

$$\rho_{nn}=\left(\frac{A}{C}\right)^n(U(x,y))^n\rho_{00}\prod_{j=0}^{n}\left[\left[1-\frac{j}{4}\frac{B}{A}\left\{1+\frac{\gamma_{ab}^2}{\gamma_{ab}^2+\Delta^2}\right.\right.\right.$$
$$\left.\left.\left.+\frac{y}{U(x,y)}\left[\frac{1}{2}\left(\frac{\partial V}{\partial x}-\frac{\partial U}{\partial y}\right)+\frac{\gamma_{ab}^2}{\Delta^2+\gamma_{ab}^2}\frac{V(x,y)}{x}\right]\right\}\right]\right]. \tag{7}$$

3. Operational characteristics

It is easy to predict the threshold condition for laser action from equation (7) because a peak for the photon distribution occurs if $(A/C)U(x,y)\geqslant 1$, so that the threshold condition is written down as:

$$AU(x,y)\geqslant C. \tag{8}$$

This equation shows that the threshold condition is dependent on y. Even when γ_{ab} is assumed to be smaller than ku, the relationship is:

$$A\exp(-x^2)\left(1-\frac{2y}{\sqrt{\pi}}\right)\geqslant C.$$

Figure 1 displays the dependence of the threshold condition on γ_{ab}/ku as described by equation (8).

The distribution ρ_{nn} shows a peak for operation above threshold. The actual position of the peak at $n=\bar{n}$ can be computed from equation (7) with the condition

$$\sum_n \rho_{nn}=1$$

However, an approximate value of $\bar{n}$ is calculated assuming $\rho_{\bar{n}-1,\bar{n}-1}=\rho_{\bar{n}\bar{n}}$ giving

$$\bar{n}=4\frac{A}{B}\left(\frac{U(x,y)-C/A}{\{1+[\gamma_{ab}^2/(\gamma_{ab}^2+\Delta^2)]\}(U(x,y)+2y/\sqrt{\pi})}\right) \tag{9}$$

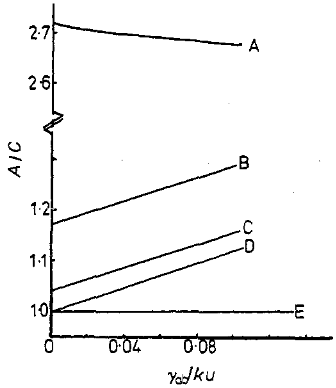

Figure 1. Dependence of A/C on γ_{ab}/ku. A, $\Delta = ku$; B, $\Delta = 0{\cdot}4\,ku$; C, $\Delta = 0{\cdot}2\,ku$; D, $\Delta = 0$; E, Riska–Stenholm results ($\Delta = 0$).

where the approximation $y \ll 1$ is utilized. A comparison between equation (9) and the expression for $\bar{n}$ given by Riska and Stenholm (1970b) shows that on substituting e^{-x^2} for $u(x, y)$ and neglecting y completely the present expression reduces to that of Riska and Stenholm (1970b).

The approximate value for the halfwidth of the distribution is derived assuming $\rho_{\bar{n}+k,\bar{n}+k} = \frac{1}{2}\rho_{\bar{n}\bar{n}}$ and is given by

$$k^2 = \frac{\bar{n}}{A/CU(x, y) - 1}. \tag{10}$$

It should be noted that equation (9), rewritten as $(B/A)\bar{n}$ gives the expression for the intensity parameter (Lamb 1964) for the laser.

4. Discussion

Equation (8) predicts a threshold condition which is significantly higher, even for $\gamma_{ab}/ku = 0{\cdot}1$ compared to the results of both Riska and Stenholm (1970a, b) and Scully and Lamb (1967). In fact, if the condition $\gamma_{ab}/ku \to 0$ is assumed, equation (8) reduces to the Riska–Stenholm (1970b) expression for the threshold condition. We have treated the loss mechanism in the same way as Riska and Stenholm (1970a) assuming $\gamma_{\alpha\beta} \gg ku$. However, the linear loss of the system can be calculated exactly and we find

$$\frac{A}{C} = \frac{\gamma_\beta}{\gamma_a}\frac{r_a}{r_\beta}\frac{1}{w(i\gamma_{\gamma\beta}/ku)}. \tag{11}$$

For a system with $r_a \gg r_\beta$ and $\gamma_\beta > \gamma_a$, the condition $\gamma_{\alpha\beta} \gg ku$ is unnecessary. If the system satisfies the condition $\gamma_{\alpha\beta} \ll ku$, we have

$$A/C = N_a/N_\beta$$

where N_η is is the population of the level η. Equation (11) reduces to the Riska-Stenholm result in the approximation $\gamma_{\alpha\beta} \gg ku$.

The value of $\bar{n}$, as given by equation (9), is considerably lower than the previously accepted values (Riska and Stenholm 1970b). For example, at $B/A=0{\cdot}005$ and $A/C=1{\cdot}2$ at $\Delta=0$ and $\gamma_{ab}/ku=0{\cdot}1$; $\bar{n}$ is 27 whereas the corresponding value given by Riska–Stenholm (1970b) is 67. Even at the same level of excitation above threshold ($U(x,y)A/C$ (present work) $=A/C$ (Riska–Stenholm)) the present value of $\bar{n}$ is 55. The nature of expansion used in equation (9) introduces at best a 2% error when y is taken to be $0{\cdot}1$. Even if the term $2y/\sqrt{\pi}$ is neglected in the denominator of equation (9) the value of $\bar{n}$ comes out to be 62 instead of 55 under the condition mentioned above.

For purpose of comparison with existing results we rewrite the equation (7) for $y \ll 1$:

$$\rho_{nn}=(A/C)^n\left[\left(1-\frac{2y}{\sqrt{\pi}}\right)e^{-x^2}\right]^n\rho_{0,0}\prod_{j=0}^{n}\left\{1-\frac{jB}{4A}\left[\left(1+\frac{\gamma_{ab}^2}{\gamma_{ab}^2+\Delta^2}\right)\left(1+\frac{2y}{\sqrt{\pi}}\right)\right]\right\}. \tag{12}$$

Figure 2 compares the results of present calculation with that of Riska–Stenholm (1970b) for the same level of excitation.

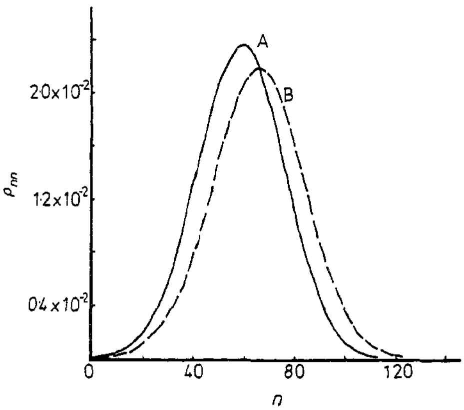

Figure 2. The photon distribution from the present work is compared with the result of Riska and Stenholm for the same level of excitation. A, present work; B, Riska–Stenholm work. $\Delta=0$, $y=0{\cdot}1$, $A/C\,U(0,y)=1{\cdot}2$.

Figure 3 shows the distribution curve for $\Delta=0$ when an exact calculation based on the non-perturbative approach (Riska and Stenholm 1970a) is made at resonance (Mohanty and Nayak 1976). Unlike Riska and Stenholm's (1970a, b) work, where there was a 30% discrepancy, the discrepancy here is much lower, which may mean that the discrepancy in Riska and Stenholm's work was due more to the assumption $\gamma_{ab}/ku\to 0$ than to the perturbation approximation. This means that the results of this calculation are correct for a low level of excitation when the higher order perturbations can be safely neglected.

It must be understood that while giving an insight into gas laser operating conditions, this theory is not capable of depicting the exact physical situation. As we do not deal with the exact nature of γ_{ab}, a quantitative evaluation of the pressure effect on a gas laser is out of the scope of present discussion. Considerable work exists on this aspect of laser operation (see, as a recent example, Stenholm 1970 and his earlier references) and a simple substitution of γ_{ab} gives only a qualitative picture.

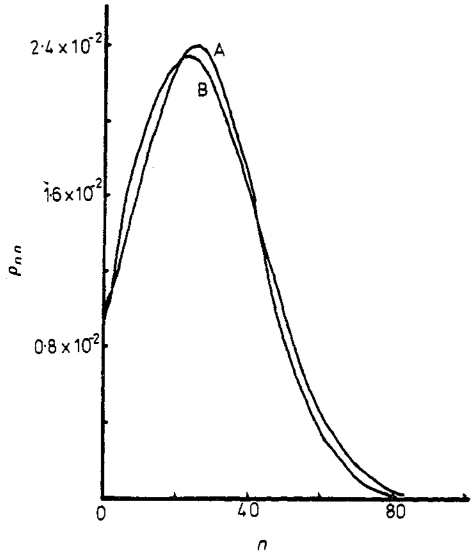

Figure 3. The approximate photon distribution function (A) is compared with the exact one at resonance (B). $\Delta=0$, $y=0\cdot1$, $A/C=1\cdot2$.

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